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On Best Approximation of the Monomials on the Unit Ball of $\mathbb{R}^r$

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Bos and Liang have separately proved that the Kergin interpolants with respect to distinguished nodes on the unit disk are best approximations of the monomials in the infinity norm. These results are extended by characterizing the nodes as solutions of a system of nonlinear equations. Thus, it is possible to get all nodes with this property on the unit disk and it is likewise possible to lift results to the ball in the sense of approximation in the mean. © 1998 Academic Press

1. INTRODUCTION

Bos [1] and Liang [5] proved that Kergin interpolants $\mathscr{K}(x^m)$ to the (multivariate) monomials x^m, $m=(m_1, m_2) \in \mathbb{N}_0^2$, are best approximations of the monomials on the unit disk B^2 in the infinity norm, provided that certain equally spaced points on the unit circle S^1 are taken as interpolation nodes. For these nodes $\|x^m - \mathscr{K}(x^m)\|_\infty = 2^{1-|m|}$, $x \in B^2$.

The question is whether there exist other interpolation nodes for which the corresponding Kergin interpolants are best approximations of the monomials on the disk. Furthermore, one can ask how to calculate such nodes and one can reflect upon lifting the problem of best approximation on the disk to higher dimensions.

These questions are answered by Theorem 4.1 below which gives a characterization of the desired nodes.

First, in Section 2 an explicit representation of the Kergin interpolants to the monomials and their coefficient is given which is important for further results. In Section 3 the nodal systems which yield best approximating Kergin interpolants are characterized and in Sections 4 and 5 this characterization is used to extend the result of Bos and Liang to further nodes.

In this particle $\mathbb{P}_\mu^r$ denotes the linear space of polynomials of degree $\leqslant \mu$ in r variables and $\overset{*}{\mathbb{P}}{}_\mu^r$ is defined to be the linear space of homogeneous polynomials of degree μ.

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2. KERGIN INTERPOLANTS TO THE MONOMIALS

In 1978, Kergin [2, 3] proposed an operator $C^{\mu}(\mathbb{R}^r) \to \mathbb{P}^r_{\mu}$ which only needs $\mu+1$ nodes to interpolate a function $f \in C^{\mu}(\mathbb{R}^r)$ uniquely, i.e. interpolation is done at far fewer nodes that is given by the dimension of $\mathbb{P}^r_{\mu}$.

For the monomials a representation of the Kergin interpolants is known which, in contrast to the Kergin interpolants to arbitrary functions, can be evaluated numerically.

Using *elementary symmetric polynomials* that are defined for $t_0, ..., t_n \in \mathbb{R}$ by $s_j(t_0, ..., t_n) := \sum_{0 \leqslant i_1 < \cdots < i_j \leqslant n} t_{i_1} \cdot t_{i_2} \cdots t_{i_j}$, $j \in \{1, ..., n+1\}$ and $s_0(t_0, ..., t_n) := 1$ (see e.g. Kostrikin [4, p. 276]), the Kergin operator for the monomials has the following representation.

THEOREM 2.1. (Kergin [2, 3]). *Let $r \in \mathbb{N}$, $\mu \in \mathbb{N}_0$, and $x_0, ..., x_\mu \in \mathbb{R}^r$. Let $\mathscr{K}$ be the Kergin operator with respect to the nodes $x_0, ..., x_\mu$. Then for $m \in \mathbb{N}^r_0$, $|m| = \mu+1$,*

$$\mathscr{K}\left(\binom{\mu+1}{m} x^m\right) = \sum_{\substack{k \in \mathbb{N}^r_0 \\ 0 < k \leqslant m}} \binom{|m-k|}{m-k} \alpha_k x^{m-k},$$

where α_k is the coefficient of $y^k = y_1^{k_1} \cdots y_r^{k_r}$ in $(-1)^{|k|-1} s_{|k|}(y \cdot x_0, ..., y \cdot x_\mu)$, $|k| \in \{1, ..., \mu+1\}$, and $y \in \mathbb{R}^r$. For $|k| \notin \{1, ..., \mu+1\}$ let $\alpha_k := 0$.

For $m \in \mathbb{Z}^r$, $|m| = \mu$, the *multinomial coefficient* is defined here by

$$\binom{\mu}{m} = \frac{\mu!}{m_1! \cdot m_2! \cdots m_r!} \quad \text{and} \quad \binom{\mu}{m} := 0 \qquad \text{for} \quad m \in \mathbb{Z}^r, m \ngeqslant 0.$$

The coefficients α_k play an important role in obtaining a characterization of suitable interpolation nodes. Therefore, an explicit representation of the α_k—as developed in the author's dissertation [6]—shall be given here.

First, some notation is necessary.

$$\left.\begin{array}{l} \text{For } k = (k_1, ..., k_r) \in \mathbb{N}^r_0 \text{ with } 1 \leqslant |k| \leqslant \mu+1 \\ \text{let the index family } I_k \text{ be defined by} \\ I_k := \{\underbrace{1, ..., 1}_{k_4}, \underbrace{2, ..., 2}_{k_2}, ..., \underbrace{r, ..., r}_{k_r}\}. \\ \tau(I_k) \text{ denotes the set of index families that} \\ \text{are obtained from } I_k \text{ by permuting its elements.} \end{array}\right\} \qquad (1)$$

THEOREM 2.2. *Let $\mu \in \mathbb{N}_0$. For $k = (k_1, ..., k_r) \in \mathbb{N}_0^r$, $1 \leqslant |k| \leqslant \mu + 1$, let I_k and $\tau(I_k)$ be defined as in* (1). *Let $\mathscr{K}(x^m)$, $m \in \mathbb{N}_0^r$, $|m| = \mu + 1$, be the Kergin interpolant with respect to the nodes $x_0, ..., x_\mu \in \mathbb{R}^r$ and let α_k be its coefficients. Then*

$$\alpha_k = (-1)^{|k|-1} \cdot \sum_{\substack{0 \leqslant i_1 < \cdots < i_{|k|} \leqslant \mu \\ (l_1, ..., l_{|k|}) \in \tau(I_k)}} x_{i_1, l_1} \cdot x_{i_2, l_2} \cdots x_{i_{|k|}, l_{|k|}}. \tag{2}$$

(*The second indices $l_1, ..., l_{|k|}$ denote the components of the nodes $x_{i_1}, ..., x_{i_{|k|}}$.*)

3. IDENTIFICATION OF CERTAIN POLYNOMIAL FAMILIES AS KERGIN REMAINDERS

Reimer in [9, Chap. 9] introduces multivariate polynomials $A_m \in \mathbb{P}^r_{\mu+1}$, $m \in \mathbb{N}_0^r$, $|m| = \mu + 1$, which are generated by certain univariate polynomials $Q_{\mu+1} \in \mathbb{P}^1_{\mu+1} \backslash \mathbb{P}^1_\mu$, where $Q_{\mu+1}$ is only required to fulfill $Q_{\mu+1}(-\xi) = (-1)^{\mu+1} Q_{\mu+1}(\xi)$ for $\xi \in \mathbb{R}$.

$Q_{\mu+1}$ is a generating function of A_m if

$$|t|^{\mu+1} Q_{\mu+1}\left(\frac{tx}{|t|}\right) = \sum_{|m| = \mu+1} A_m(x)\, t^m \tag{3}$$

for $x, t \in \mathbb{R}^r$. For $Q_{\mu+1}(\xi) = \sum_{\nu=0}^{\lfloor (\mu+1)/2 \rfloor} a_{\mu+1-2\nu} \xi^{\mu+1-2\nu}$ it then follows that

$$A_m(x) = \sum_{|n| \leqslant \lfloor (\mu+1)/2 \rfloor} a_{\mu+1-2|n|} \binom{\mu+1-2|n|}{m-2n} \binom{|n|}{n} x^{m-2n}. \tag{4}$$

Let $\mathscr{K}(x^m)$ denote the Kergin interpolant to the monomial x^m with respect to the nodes $x_0, ..., x_\mu \in \mathbb{R}^r$. For the equidistant nodes of Liang on S^1 there is

$$A_m(x) = a_{\mu+1} \binom{\mu+1}{m} (x^m - \mathscr{K}(x^m)), \tag{5}$$

with $a_{\mu+1} = 2^{|m|-1}$ the leading coefficients of the Tschebyscheff polynomials of the first kind $T_{\mu+1}$.

According to a result of Reimer [7] the polynomials A_m are (for $r = 2$) the polynomials of least deviation from zero with respect to the maximum norm, thus yielding that the Kergin interpolants to the equidistant nodes of Liang are best approximations to the monomials (see [5]).

The aim now will be to determine interpolation nodes on S^{r-1}, $r \geqslant 2$, such that relation (5) is valid with respect to the maximum norm or (for $r \geqslant 3$) at least with respect to the L^2-norm (see Section 5).

Relation (5) is equivalent to

$$\mathscr{K}(x^m) = x^m - a_{\mu+1}^{-1} \binom{\mu+1}{m}^{-1} A_m(x) \tag{6}$$

which results in a characterization of the corresponding nodes.

THEOREM 3.1. *Let* $m \in \mathbb{N}_0^r$, $|m| = \mu + 1$. *Let the polynomials* A_m *be generated as in* (3). *The Kergin interpolant* $\mathscr{K}(x^m)$, *with respect to the nodes* $x_0, ..., x_\mu$, *satisfies relation* (6) *if and only if the coefficients* $\alpha_k = \alpha_k(x_0, ..., x_\mu)$ *of* $\mathscr{K}(x^m)$, $k \in \mathbb{N}_0^r$, $0 \neq k \leqslant m$, *fulfill the following equations*:

$$\left.\begin{aligned} &\alpha_k = 0 && \text{for} \quad k \neq 2n, \quad k \leqslant m, \\ &\alpha_k = -\binom{|n|}{n} \frac{a_{\mu+1-2|n|}}{a_{\mu+1}} && \text{for} \quad k = 2n, \quad 1 \leqslant |n| \leqslant \left\lfloor \frac{\mu+1}{2} \right\rfloor. \end{aligned}\right\} \tag{7}$$

Proof. By the use of (4), a comparison of (6) and of the representation of $\mathscr{K}(x^m)$ given in Theorem 2.1 yields (7). ∎

4. RESULTS ON B^2

We are now able to state an extension of the result of Bos and Liang.

THEOREM 4.1. *Let* $r = 2$, $Q_{\mu+1} = T_{\mu+1}$, $\mu \in \mathbb{N}_0$, *and let* $m \in \mathbb{N}_0^2$, $|m| = \mu + 1$. *If the nodes* $x_0, ..., x_\mu \in S^1$ *satisfy the system* (7), *resulting by inserting* (2), *then the corresponding Kergin interpolant* $\mathscr{K}(x^m)$ *is a best approximation to the monomial* x^m, *i.e.*, $\|x^m - \mathscr{K}(x^m)\|_\infty = 2^{1-|m|}$, $x \in B^2$.

Proof. The nodal systems of Liang are solutions of (7) for $Q_{\mu+1} = T_{\mu+1}$ with the parameters $a_{\mu+1-2|n|}$, $0 \leqslant |n| \leqslant \lfloor (\mu+1)/2 \rfloor$ being the coefficients of the Tschebyscheff polynomials $T_{\mu+1}$. The statement then follows by the result of Liang [5]. For other nodal systems it follows analogously. ∎

EXAMPLE. The solution of (7) for $\mu = 2$ yields the nodes given in the following table. Tables for $\mu = 1$ and $\mu = 3$ can be found in [6]. In the table the nodes of Liang are marked by a star.

Nodes for $\mu = 2$

x^m	Nodal Systems x_0	x_1	x_2	
x_1^3	$(0, 1)$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	
	$(0, -1)$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	*
	$(0, 1)$	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	
	$(0, -1)$	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	
	$(0, 1)$	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	*
	$(0, -1)$	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	
	$(0, 1)$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	
	$(0, -1)$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	
x_2^3	Nodes for x_1^3 with interchanged components			
$x_1^2 x_2$	$(1, 0)$	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	*
	$(-1, 0)$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	*
$x_1 x_2^2$	nodes for $x_1^2 x_2$ with interchanged components			

It turns out that the only equally spaced nodes are those of Liang. In particular, there exist solutions of (7) that are not equidistributed.

5. RESULTS ON B^r, $r \geqslant 3$

According to Reimer [8], it is impossible in the case $r \geqslant 3$ to generate polynomials of least deviation from zero with respect to the maximum norm by rational functions. Therefore, the results of case $r=2$ cannot be lifted completely to higher dimensions. Nevertheless, it is possible to obtain similar results with respect to $\|\cdot\|_2$.

Let p, q be arbitrary polynomials. Define the semi-inner product

$$\langle p, q\rangle := \int_{S^{r-1}} p(x)\, q(x)\, dx.$$

Now the problem of best approximation can be investigated with respect to the seminorm $\|p\|_2 := \sqrt{\langle p, p\rangle}$.

Let $A_m \in \mathbb{P}^r_{\mu+1}$, $m \in \mathbb{N}^r_0$, $|m| = \mu+1$, be generated as in (3). Homogenization of the polynomials A_m leads to $\overset{*}{A}_m(x) := |x|^{\mu+1} A_m(x/|x|)$, $x \in \mathbb{R}^r$, with $\overset{*}{A}_m(x) = A_m(x)$ for $x \in S^{r-1}$.

Let the coefficient functionals $c_m\colon \mathbb{P}^r_{\mu+1} \to \mathbb{R}$, $m \in \mathbb{N}^r_0$, be defined by $P(x) = \sum_{|m| \leqslant \mu+1} c_m(P)\, x^m$ for $P \in \mathbb{P}^r_{\mu+1}$ (Reimer [9, p. 32]) and let $\overset{*}{c}_m$ denote the restriction of the coefficient onto $\overset{*}{\mathbb{P}}{}^r_{\mu+1}$ with the representer $\overset{*}{A}_m$, i.e., $\overset{*}{c}_m(P) = \langle P, \overset{*}{A}_m\rangle$ for arbitrary $P \in \overset{*}{\mathbb{P}}{}^r_{\mu+1}$. Then the following theorem is valid for $\overset{*}{A}_m$.

Theorem 5.1. (Reimer [9, Theorem 9.4]). *Let $r \geqslant 2$, $m \in \mathbb{N}^r_0$, $|m| = \mu+1$, $\mu \in \mathbb{N}_0$. Then $|\overset{*}{c}_m(P)| \leqslant \overset{*}{c}_m(\overset{*}{A}_m)$ holds for arbitrary $P \in \overset{*}{\mathbb{P}}{}^r_{\mu+1}$ with $\|P\|_2 \leqslant \|\overset{*}{A}_m\|_2$. Conversely, $P \in \overset{*}{\mathbb{P}}{}^r_{\mu+1}$ and $|\overset{*}{c}_m(P)| = \overset{*}{c}_m(\overset{*}{A}_m)$ implies $\|P\|_2 \geqslant \|\overset{*}{A}_m\|_2$. Equality occurs if and only if $P = \pm \overset{*}{A}_m$.*

In the following lemma the notation $n \equiv m$ means that $n_j = m_j \bmod(2)$ for $j = 1, 2, ..., r$ and the $a_{|n|}$ denote the coefficients of $Q_{\mu+1}$.

Lemma 5.2 (Reimer [9, p. 76]). *Let $r \geqslant 3$, $m \in \mathbb{N}^r_0$, $|m| = \mu+1$. Then*

$$\|\overset{*}{A}_m\|_2^2 = \sum_{\substack{n \leqslant m \\ n \equiv m}} \binom{|n|}{n} \binom{\frac{|m-n|}{2}}{\frac{m-n}{2}}^2 a_{|n|}.$$

From this a result similar to Theorem 4.1 can be derived.

Theorem 5.3. *Let $r \in \mathbb{N}$, $r \geqslant 3$, and $m \in \mathbb{N}^r_0$, $|m| = \mu+1$, $\mu \in \mathbb{N}_0$. If the nodes $x_0, ..., x_\mu \in S^{r-1}$ satisfy the nonlinear system (7), resulting from* *inserting* (2), *then the corresponding Kergin interpolant* $\mathcal{K}(x^m)$ *is a best approximation of the monomial* x^m *on* S^{r-1} *with respect to* $\|\cdot\|_2$ *and*

$$\|x^m - \mathcal{K}(x^m)\|_2 = a_{\mu+1}^{-1} \binom{\mu+1}{m}^{-1} \cdot \left(\sum_{\substack{n \leqslant m \\ n \equiv m}} \binom{|n|}{n} \binom{\frac{|m-n|}{2}}{\frac{m-n}{2}}^2 a_{|n|} \right)^{1/2}, \qquad x \in S^{r-1}.$$

Proof. $A_m(x) = a_{\mu+1}\binom{\mu+1}{m}(x^m - \mathcal{K}(x^m))$ if the interpolation nodes fulfill (7). Let $P \in \overset{*}{\mathbb{P}}{}^r_{\mu+1}$ be arbitrary with $|\overset{*}{c}_m(P)| = \overset{*}{c}_m(\overset{*}{A}_m) = a_{\mu+1}\binom{\mu+1}{m}$. Now application of Theorem 5.1 gives

$$\|P\|_2 \geqslant \|\overset{*}{A}_m\|_2 = |a_{\mu+1}| \binom{\mu+1}{m} \|x^m - \mathcal{K}(x^m)\|_2, \qquad x \in S^{r-1}.$$

Lemma 5.2 concludes the proof. ∎

Remark. Numerical experiments in case $r = 3$ show that there exist solutions of (7) corresponding with the situation of Theorem 5.3. For instance, in the case of $\mu = 3$, the vertices of certain tetrahedra circumscribed by the unit sphere S^2 are solutions. In this case ($r = 3$, $\mu = 3$) it turns out that

$$Q_4(\xi) = \frac{315}{8} \xi^4 - \frac{105}{4} \xi^2 + \frac{15}{8}$$

is the unique possible choice for the generating function in (3). Apart from a constant factor this is just the reproducing kernel of $\mathbb{P}_4^{*3}$.

Tables of nodes for all multiindices $m \in \mathbb{N}_0^3$, $|m| = 4$, can be found in [6].

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